

Nonlinear hydrodynamic stability

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 (July 28, 1997)

The variational principle of V. I. Arnold [J. Appl. Math. Mech. Vol. 29, P. 1002 (1965)] is extended to the general conservative inhomogeneous, compressible, and conducting fluid. The concept of iso-vortical flows is generalized to an “invariant foliation” of the phase-space. The foliation, which may or may not correspond to explicit conservation laws, is derived from the equations of motion and used for Lyapunov stability. A nonlinear three-dimensional (magneto-) hydrodynamic stability criterion is formulated.

The standard approach to hydrodynamic stability involves linearization about an equilibrium flow in order to solve for eigenfrequencies [1,2] or establish a Lyapunov stability criterion for the linearized system [3]. A variety of linear variational principles were developed for both neutral fluids [4] and magnetohydrodynamics (MHD) [5–7], in which the stability criterion is expressed in terms of a positive definite quadratic form. It is well known that the linearized stability does not guarantee the true (Lyapunov) stability, such as in the toy system $du/dt = u^2$, whose equilibrium $u = 0$ is linearly stable.

Nonlinear stability is guaranteed by the presence of an integral of motion, for example, the energy H , which assumes a non-degenerate extremum (a minimum or a maximum) subject to the conservation of any other integrals of motion, for example, Casimir invariants [8]. The possibility to write explicitly a full infinite set of integrals is mostly limited to two-dimensional systems. By *explicit* we mean an integral of motion which can be written in terms of the physical fields of velocity, density, etc., in a way which does not require the solution of the equations of the motion. In three dimensions, such integrals are scarce. For example, the Euler equation

$$\partial_t \omega = \nabla \times (\mathbf{v} \times \omega), \quad \omega = \nabla \times \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0, \quad (1)$$

conserves explicitly only the energy H and the helicity I :

$$H = \int \frac{\mathbf{v}^2}{2}, \quad I = \int \mathbf{v} \cdot \omega. \quad (2)$$

(Here and below, unless specified, all integrals are understood over the domain occupied by the fluid. An appropriate conservative boundary condition, such as zero normal velocity, is implied.) In addition to the explicit integrals (2), there is also the infinity of Kelvin invariants,

$$I_\gamma = \oint_\gamma \mathbf{v} \cdot d\ell \equiv \int_\gamma \omega \cdot d\mathbf{S} = \text{const}, \quad (3)$$

expressing the velocity circulation around (or the vorticity flux through) any closed contour $\gamma(t)$ moving with the fluid velocity \mathbf{v} . Integrals (3) are *implicit* in the sense that their definition involves contours γ whose motion must be solved from Eq. (1). Although there is no apparent way of incorporating integrals like (3) in a Lyapunov functional, Arnold [9] proposed that the conservation of all vorticity integrals (3) has the geometrical meaning of confining the system to an “iso-vortical sheet” in the infinite-dimensional phase space. Different sets of initial vorticity integrals specify different sheets such that the whole phase space is “foliated,” as if by iso-surfaces of an integral of motion (Figure 1).

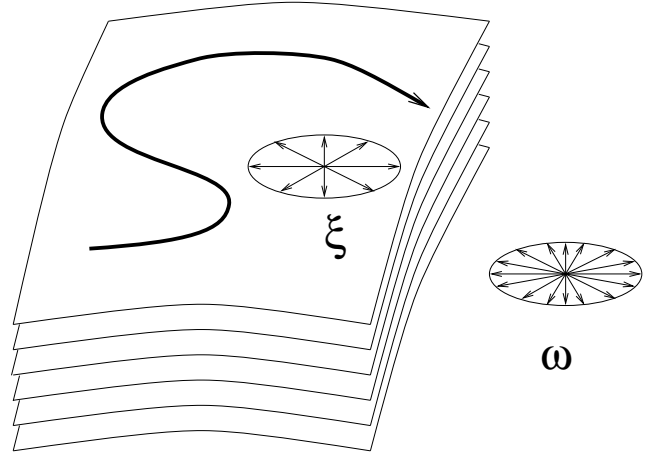


FIG. 1. The infinite-dimensional phase space of all incompressible fluid flows, $\omega(\mathbf{x})$, is foliated by iso-vortical invariant sheets parameterized by the displacement function $\xi(\mathbf{x})$. Each sheet is an infinite-dimensional subspace of ω . The dynamics keeps an orbit on a sheet.

The usefulness of the foliation for stability is due to the local explicit parameterization of the iso-vortical sheets by an incompressible “displacement” $\xi(\mathbf{x})$, such that vorticity fields sharing the sheet with the reference flow $\omega_0(\mathbf{x})$ are written $\omega = \omega_0 + \delta\omega_0 + \frac{1}{2}\delta^2\omega_0 + \dots$, where

$$\delta\omega = \nabla \times (\xi \times \omega). \quad (4)$$

The iso-vortical variation operator δ derives from the “modified dynamics” $\partial_t \omega = \nabla \times (\partial_t \xi \times \omega)$, $\nabla \cdot \xi = 0$, in

which the vorticity is incompressibly advected in a way similar to the Euler equation (1), but by the velocity field $\partial_t \boldsymbol{\xi}$ entirely unrelated to the actual flow \mathbf{v} . Since the conservation of the vorticity integrals (3) is independent of the relation between \mathbf{v} and $\boldsymbol{\omega}$, Eq. (4) follows.

Arnold variation (4) makes the Hamiltonian H stationary if and only if the flow \mathbf{v} is in equilibrium. Then the second energy variation,

$$\delta^2 H = \int (\delta \mathbf{v})^2 - \boldsymbol{\xi} \times \boldsymbol{\omega} \cdot \nabla \times (\boldsymbol{\xi} \times \mathbf{v}), \quad (5)$$

if definite for all incompressible $\boldsymbol{\xi}$, guarantees that the equilibrium is stable [9,10].

Given this long introduction, we briefly report on a generalization of the Arnold method in two important ways. Firstly, our fluid equations (6)–(9) include compressibility, varying entropy and also magnetic field, but still no dissipation. In such a general formulation, it is difficult to write all integrals generalizing Eq. (3). Therefore, and secondly, an analog of the iso-vortical variation is formally derived from the dynamics, without regard to either explicit or implicit integrals of motion. A new outcome of this procedure is a variational principle for ideal MHD stability with fluid flow, a long-standing plasma physics problem [6,7,11,3].

We consider the following hydrodynamic equations for an inviscid, ideally conducting fluid:

$$\rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p(\rho, s) + \mathbf{j} \times \mathbf{B} - \rho \nabla \phi, \quad (6)$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (7)$$

$$\partial_t s + \mathbf{v} \cdot \nabla s = 0, \quad (8)$$

$$\partial_t \rho + \nabla \cdot \rho \mathbf{v} = 0. \quad (9)$$

Here p is the fluid pressure, ρ the density, s the entropy, ϕ the external gravitational potential, \mathbf{B} the magnetic field, and $\mathbf{j} = \nabla \times \mathbf{B}$ the electric current. The fluid flow conserves the energy

$$H = \int \rho \left[\frac{\mathbf{v}^2}{2} + \epsilon(\rho, s) + \phi \right] + \frac{\mathbf{B}^2}{2}, \quad (10)$$

where ϵ is the specific internal energy defined by the standard thermodynamic relation

$$d\epsilon = T ds - p d(1/\rho). \quad (11)$$

The varying entropy and the Lorentz force in Eq. (6) break the “frozen-in” law for the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, and instead of (1) we now have

$$\partial_t \boldsymbol{\omega} = \nabla \times \left(\mathbf{v} \times \boldsymbol{\omega} + \mathbf{j} \times \frac{\mathbf{B}}{\rho} + \int^\rho \frac{\partial_s p d\rho}{\rho^2} \nabla s \right). \quad (12)$$

By analogy with the iso-vortical variation (4), one can introduce modified dynamics for Eqs. (6)–(9) in many different ways [12]. Our choice is dictated by the desire to

have a zero energy variation for equilibrium flows. Upon a dozen of attempts, the following procedure works to our satisfaction: We replace $\mathbf{v} \rightarrow \partial_t \boldsymbol{\xi}$ in Eqs. (7)–(9) and (12). In (12), we also write $\mathbf{j} \rightarrow \partial_t \boldsymbol{\eta}$ ($\nabla \cdot \boldsymbol{\eta} = 0$) and replace the integral by a scalar $\partial_t \alpha$. The result is the *generalized iso-vortical variation*,

$$\delta \mathbf{v} = \boldsymbol{\xi} \times \boldsymbol{\omega} + \boldsymbol{\eta} \times \frac{\mathbf{B}}{\rho} + \alpha \nabla s + \nabla \beta, \quad \boldsymbol{\eta} = \nabla \times \boldsymbol{\zeta}, \quad (13)$$

$$\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}), \quad \delta s = -\boldsymbol{\xi} \cdot \nabla s, \quad \delta \rho = -\nabla \cdot \rho \boldsymbol{\xi}, \quad (14)$$

which depends on two arbitrary vectors $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ and two arbitrary scalars α and β .

Although variation (13)–(14) conserves magnetic flux and entropy integrals (which are well known and not written here), it is unclear what other conservation laws, if any, are accounted for by this variation. Nevertheless, the derivation above clearly implies that the phase-space sheets parameterized by $(\boldsymbol{\xi}, \boldsymbol{\zeta}, \alpha, \beta)$ are invariant sheets, which can be interpreted as iso-surfaces of some integrals of motion and thus used for stability analysis. In the limit of zero magnetic field and constant entropy, variation (13) reduces to Arnold’s iso-vortical variation (4).

The number of arbitrary functions in the variation (13)–(14) by no accident equals the number of dynamical equations (6)–(9). Upon varying the energy (10) and using Eq. (11), a few integrations by parts yield

$$\delta H = \int \boldsymbol{\xi} \cdot (\rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p - \mathbf{j} \times \mathbf{B} + \rho \nabla \phi) - \boldsymbol{\zeta} \cdot \nabla \times (\mathbf{v} \times \mathbf{B}) + \alpha \rho \mathbf{v} \cdot \nabla s - \beta \nabla \cdot \rho \mathbf{v} \quad (15)$$

By design, the condition that $\delta H = 0$ for all $(\boldsymbol{\xi}, \boldsymbol{\zeta}, \alpha, \beta)$ is equivalent to an equilibrium solution of Eqs. (6)–(9).

The second variation of velocity,

$$\delta^2 \mathbf{v} = \boldsymbol{\xi} \times \delta \boldsymbol{\omega} + \boldsymbol{\eta} \times \delta \frac{\mathbf{B}}{\rho} + \alpha \nabla \delta s + \nabla \beta, \quad (16)$$

and similar expressions for $\delta^2(\mathbf{B}, s, \rho)$ are now used to calculate the second energy variation:

$$\delta^2 H = \int \delta^2 \left(\rho \epsilon + \frac{\mathbf{B}^2}{2} \right) + \left(\phi + \frac{\mathbf{v}^2}{2} \right) \delta^2 \rho + \rho (\delta \mathbf{v})^2 + \rho \mathbf{v} \cdot \delta^2 \mathbf{v} + 2 \delta \rho \mathbf{v} \cdot \delta \mathbf{v}. \quad (17)$$

Equations (13)–(14) define $\delta^2 H$ as a functional of $(\boldsymbol{\xi}, \boldsymbol{\zeta}, \alpha, \beta)$. Two comments regarding the form of $\delta^2 H$ are in order.

First, the suspicious linear term $\nabla \beta$ in the second velocity variation (16) is multiplied by an incompressible $\rho \mathbf{v}$ in Eq. (17) and thus vanishes upon integration by parts. So, as it should, $\delta^2 H$ is a *quadratic* functional of the independent variables $(\boldsymbol{\xi}, \boldsymbol{\zeta}, \alpha, \beta)$.

Second, the integrand of (17) can be written as a quadratic polynomial of α with the coefficient $\rho(\nabla s)^2$ in front of α^2 . Therefore, the definite sign of $\delta^2 H$ can be

only positive, and, for this, it is necessary and sufficient that the reduced quadratic form of (ξ, ζ, β) ,

$$W \equiv \min_{\alpha} \delta^2 H = \int \delta^2 \left(\rho \epsilon + \frac{\mathbf{B}^2}{2} \right) + \left(\phi + \frac{\mathbf{v}^2}{2} \right) \delta^2 \rho - \rho (\mathbf{n} \cdot \delta'' \mathbf{v})^2 + \delta' \mathbf{v} \cdot (\rho \delta'' \mathbf{v} + \mathbf{v} \delta \rho) + \rho \mathbf{v} \cdot \boldsymbol{\eta} \times \delta \frac{\mathbf{B}}{\rho}, \quad (18)$$

be also positive definite. Here $\mathbf{n} = \nabla s / |\nabla s|$, and the space-saving notation is introduced,

$$\delta' \mathbf{v} \equiv \boldsymbol{\xi} \times \boldsymbol{\omega} + \boldsymbol{\eta} \times \mathbf{B} / \rho + \nabla \beta, \quad \delta'' \mathbf{v} \equiv \delta' \mathbf{v} + \boldsymbol{\xi} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \boldsymbol{\xi}.$$

No further “simple” minimization of Eq. (18) is possible.

To make Eq. (18) more explicit, the internal energy variation can be transformed as

$$\int \delta^2 (\rho \epsilon) = \int \nabla \cdot \boldsymbol{\xi} (\rho \partial_{\rho} p \nabla \cdot \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla p), \quad (19)$$

and the magnetic energy variation

$$\int \delta^2 \frac{\mathbf{B}^2}{2} = \int \delta \mathbf{B} \cdot (\delta \mathbf{B} - \boldsymbol{\xi} \times \mathbf{j}). \quad (20)$$

Also, the second density variation is $\delta^2 \rho = \nabla \cdot (\boldsymbol{\xi} \nabla \cdot \rho \boldsymbol{\xi})$.

Equation (18) is the main result of this paper. It can be used to establish nonlinear stability of general MHD equilibria with fluid flow, for example numerically, by testing the sign of W for sets of seven scalar trial functions (ξ, ζ, β) . Since all known explicit and implicit integrals of motion are incorporated in our scheme (the possibly conserved linear and angular momenta amount to choosing an appropriate frame of reference), *we propose that the sufficient stability criterion $W > 0$ is also necessary for the true nonlinear stability of an ideal MHD equilibrium.* This conjecture is also supported by the static limit of zero flow, $\mathbf{v} = 0$, in which our variational principle reduces to the sum of Eqs. (19) and (20), or the standard MHD energy principle [5], whose violation means a linear instability. As a by-product, we thus find that the linear stability criterion of Bernstein *et al.* [5] for static equilibria is also a nonlinear stability criterion. In a general situation with fluid flow, an indefinite W may not result in an exponential instability, but rather lead to a slower, nonlinear perturbation growth and subsequent turbulence. This scenario will be described elsewhere.

The other two limiting cases we would like to mention are (a) the hydrostatic equilibrium with $\phi = gz$ and $\mathbf{v} = \mathbf{B} = 0$ and (b) the incompressible neutral fluid with $\rho = s = \text{const}$ and $\mathbf{B} = 0$. For the former case, the condition $W > 0$ yields the well known convective stability criterion [2]: $ds/dz > 0$ and $d\rho/dz < 0$. In the Euler limit, the incompressibility is introduced by letting the sound speed $c^2 = \partial_{\rho} p$ to infinity. The minimum of Eq. (19) then implies $\nabla \cdot \boldsymbol{\xi} \rightarrow 0$ for the “most dangerous” perturbations, and further minimization of (17) with respect to β results

in $\nabla \cdot \delta \mathbf{v} = 0$ and the restricted Arnold criterion that Eq. (5) be *positive* definite.

Thus, in several evident limiting cases, our stability criterion reduces to the already known results. However, in the general case of flow of an inhomogeneous fluid, with or without magnetic field, it appears new. The exact mathematical meaning of the generalized isovortical variation and the status of the resulting stability criterion $W > 0$, Eq. (18), remain unclear to this author. For example, no *a priori* estimates exist for three-dimensional hydrodynamic perturbations, unlike those in two dimensions, where all Casimir integrals are explicit [10,8]. On the “physical level,” the sufficient stability criterion $W > 0$ looks rigorous, less so as a necessary one.

Acknowledgments. This work was supported by the U.S. DOE Grant No. DE-FG0388ER53275, the ONR Grant No. N00014-91-J-1127, and the NSF Grant PHY94-21318.

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 - [12] This ambiguity is present already in the Euler equation. For example, one can get a different modified dynamics by replacing $\boldsymbol{\omega}$, and not \mathbf{v} , by $\partial_t \boldsymbol{\xi}$ in Eq. (1). As a result, the variation $\delta_1 \boldsymbol{\omega} = \nabla \times (\mathbf{v} \times \boldsymbol{\xi})$ defines a new foliation of the phase space by different invariant sheets. Interestingly, the new variation δ_1 keeps energy H constant to all orders, but makes the helicity I stationary only for equilibrium flows and thus allows to study their stability by inspecting the second helicity variation $\delta_1^2 I$. This observation was made by A. V. Gruzinov.